ON A COMPUTATIONALLY EFFICIENT MICROCOMPUTER KINEMATIC ANALYSIS OF THE BASIC LINKAGE MECHANISMS

FERDINAND FREUDENSTEIN† and HOMAYOON S. MOHAMMAD BEIGI‡
Department of Mechanical Engineering, Columbia University, New York, NY 10027, U.S.A.

Abstract—Based on the algebraic correspondence between the displacement equations of the plane, spherical and skew four-bar linkages and the plane and skew slider-crank mechanisms, a highly compact, computationally efficient procedure has been developed for a microcomputer kinematic analysis of these linkages.

INTRODUCTION

The computer-aided kinematic analysis of linkage mechanisms can be implemented via large-scale, general-purpose codes (such as ADAMS, DADS, DRAM, IMP etc.) or special-purpose microcomputer codes [1]. The former typically utilize incremental numerical solutions of the loop-closure equations, while the latter are generally limited to single-loop mechanisms for which algebraic displacement equations can be formulated.

In connection with the latter group it has occurred to us that a single program applicable to many basic linkage mechanisms could be developed by utilizing the algebraic correspondence between their displacement equations. The following represents a development of these ideas.

THE ALGEBRAIC CORRESPONDENCE BETWEEN THE DISPLACEMENT EQUATIONS OF THE BASIC LINKAGE MECHANISMS

General observations

In his classic monograph [2], F. M. Dimentberg already found an algebraic correspondence between the displacement equations of the plane and spherical four-bar linkages. Considering the half-tangent form of these equations, Dimentberg showed that for any spherical four-bar linkage there exists a corresponding plane four-bar linkage such that for a given crank angle the half tangents of the output-link displacements are in constant proportion. In principle this would permit a kinematic analysis of both plane and spherical four-bar linkages by means of a single computer code, say, for the plane four-bar linkage, the transformation equations between the link lengths of the corresponding linkages being included in the code.

In the present investigation we have considered another type of correspondence between the displacement equations of five basic linkage mechanisms: the plane, spherical and skew four-bar linkages and the plane and skew slider cranks. The correspondence resides in the terms of their displacement equations expressed in half-tangent form. A generalized displacement equation is developed for all of the five mechanisms with velocities, as well as accelerations, obtainable by differentiation.

Algebraic development

The half-tangent form of the displacement equations of the basic linkages shown in Figs 1 and 2 are well known [2]. In order to clarify the algebraic correspondence between the displacement equations of the linkage mechanisms it is desirable to express the coefficients of the displacements equations as functions of the sums and differences of the four basic link lengths involved (Figs 1 and 2). In the case of the plane and skew four-bar linkages this involves factorization of terms which can be expressed as the differences of two squares, while in the case of the spherical four-bar linkage this involves conversion of the difference of two cosine terms into a product of two sine terms.

In Figs 1 and 2 the lengths of the fixed link, crank, coupler and output link are denoted by a, b, c or c* and d, respectively. The angular positions \( \phi \) and \( \psi \) of input and output link are defined by half tangents \( t = \tan \frac{1}{2} \psi \) and \( u = \tan \frac{1}{2} \phi \) or \( x \), respectively.

Letting

\[
\begin{align*}
p_1 &= (1/2) \left( a - b + c - d \right) \\
p_2 &= (1/2) \left( a - b - c + d \right) \\
p_3 &= (1/2) \left( a + b + c - d \right) \\
p_4 &= (1/2) \left( a + b - c - d \right) \\
p_5 &= (1/2) \left( a - b + c + d \right) \\
p_6 &= (1/2) \left( a - b - c + d \right) \\
p_7 &= (1/2) \left( a + b + c + d \right) \\
p_8 &= (1/2) \left( a + b - c + d \right)
\end{align*}
\]

(1)
the displacement equations of these linkages can be expressed in the following form:

\[ A_1 t^2 + A_2 u^2 + A_3 u^2 t^2 + A_4 u t + A_5 + A_6 t (u^2 + 1) + (A_7 + A_8) u + A_9 t = 0 \]  \hspace{1cm} (2)

where

\[ P = (3u^2 + 1)(A_2 + A_3 u^2 + A_4 u) + u[(A_7 + A_8) t^2 + A_4 u + A_5 + A_6 u^2] + (A_7 + A_8) t \]

\[ Q = 2(1 + t^2) \left[ 2A_1 u t + \frac{1}{2} A_4 u + A_6 u^2 \right] + (A_7 + A_8) t \]

\[ R = \frac{1 + t^2}{1 + u^2} \left[ (3u^2 + 1)[A_1 + A_3 u^2 + (A_7 + A_8) u] + t[A_4 u^2 + 1 + A_6 u + A_9] \right] \]

\[ S = 2[A_2 u + A_3 u^2 t + \frac{1}{2} A_4 u + A_6 u^2 t + \frac{1}{2}(A_7 + A_8) u] \]

and where \( s = 1 \) for a slider-crank mechanism while \( s = 0 \) for a four-bar linkage. The operator, \( \nabla_i \) \((i = 1, \ldots, 10)\) is defined in Table 1. By differentiation of the displacement equation (2) the derivatives defining the angular velocities and accelerations are readily obtained. For the four-bar linkages, for example, we find:

\[ m_1 = \frac{d^2 \psi}{d \phi^2} = \frac{1 + t^2}{1 + u^2} \times \left[ A_1 t + A_3 u t^2 + \frac{1}{2} A_4 u + \frac{1}{2} A_9 (u^2 + 1) + (A_7 + A_8) u t + \frac{1}{2} (A_9 + A_6) t^2 + \frac{1}{2} (A_7 + A_8) \right] \]  \hspace{1cm} (4)

and

\[ m_2 = \frac{1}{S} \left[ - (P m_1 + Q m_2 + R) \right] \]  \hspace{1cm} (5)

Fig. 1. The basic four-bar linkages.

Fig. 2. The basic slider-crank mechanisms.
Table 1

<table>
<thead>
<tr>
<th>i</th>
<th>Coefficient</th>
<th>Plane four-bar linkage</th>
<th>Spherical four-bar linkage</th>
<th>Skew four-bar linkage</th>
<th>Plane slider crank</th>
<th>Skew slider crank</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$A_1$</td>
<td>1</td>
<td>sin</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>$A_2$</td>
<td>1</td>
<td>sin</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>$A_3$</td>
<td>1</td>
<td>sin</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>$A_4$</td>
<td>1</td>
<td>sin</td>
<td>$\sqrt{\cos \zeta}$</td>
<td>0</td>
<td>$-4b \sin \zeta$</td>
</tr>
<tr>
<td>5</td>
<td>$A_5$</td>
<td>1</td>
<td>sin</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>$A_6$</td>
<td>0</td>
<td>0</td>
<td>$-gb \sin \zeta$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>$A_7$</td>
<td>0</td>
<td>0</td>
<td>$fd \sin \zeta$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>$A_8$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-2b$</td>
<td>$-2f \cos \zeta$</td>
</tr>
<tr>
<td>9</td>
<td>$A_9$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$-4eb$</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>$A_{10}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2b</td>
<td>$-2f \cos \zeta$</td>
</tr>
</tbody>
</table>

and similarly for the slider cranks with

$$m_i = \frac{d^2x}{d\phi^2} = \frac{1}{2}(1 + u^2)m_i$$

and

$$m_z = \frac{d^2x}{d\phi^2} = \frac{1}{2}(1 + u^2)(m_1u + m_2)$$

(7)

The computer program, written in BASIC for an IBM P/C, is given in the Appendix.

DISCUSSION

Figures 3–6 illustrate the displacements, velocities and accelerations of the output member of several linkage mechanisms according to the above program. The results were compared to those of a conventional kinematic analysis in order to verify accuracy.

The present approach could be extended to other linkages, if desired, the limitation being essentially the complexity of the mechanism. An alternative approach would be to program the kinematic anal-

![Fig. 3. Plane four-bar linkage (a) displacement analysis (b) velocity and acceleration analysis.](image)

![Fig. 4. Spherical wobble plate linkage (a) displacement analysis (b) velocity and acceleration analysis.](image)
CONCLUSION

A simple, computationally efficient computer program has been developed for the kinematic analysis of five basic linkage mechanisms on a microcomputer. Extensions to other mechanisms are conceivable as well.

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REFERENCES


APPENDIX

10 'Kinematics Analysis of Basic Linkage Mechanisms:
30    1. Plane Four-Bar Linkage.
50    2. Spherical Four-Bar Linkage.
70    3. Skew Four-Bar Linkage.
90    4. Plane Slider-Crank.
100 5. Skew Slider-Crank.
110 10
120 20
130 30
140 40
150 50
160 60
170 70
180 80
190 90
200 100
210 KEY OFF
220 PI=3.141592654*CIM=0
230 DIM DEL(7),M1(2),M11(2),MB(2),M12(2),U(2),PM(2),P(3),PSI(2),PHIB(2)
240 CLS:LOCATE 6,25:PRINT "1. Plane Four Bar"
250 LOCATE 8,25:PRINT "2. Spherical Four Bar"
260 LOCATE 10,25:PRINT "3. Skew Four Bar"
270 LOCATE 12,25:PRINT "4. Plane Slider-Crank"
280 LOCATE 14,25:PRINT "5. Skew Slider-Crank"
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170 LOCATE 14,25:PRINT "5. Skew Slider Crank"
180 LOCATE 16,25:PRINT "6. Exit"
190 LOCATE 23,25:PRINT "Enter Option: _"
200 IN115=INKEY$
210 IF IN115="$" THEN GOTO 200
220 IF VAL(IN115)>1 AND VAL(IN115)<5 THEN BARTYPE=VAL(IN115):GOTO 250
230 IF IN115="$" THEN GOTO 1360
240 SOUND 1500,1:TGOTO 200
250 IF BARTYPE<>4 THEN S=1
260 CLS:INPUT "Please enter the input crank length: ",B:BT=B
270 INPUT " enter the connecting rod length: ",C:CT=C
280 IF S=1 THEN GOTO 300
290 INPUT " enter the output crank length: ",D:DT=D
310 IF BARTYPE<>2 THEN INPUT " enter the fixed link length: ",A:DIAG=A
320 IF BARTYPE=3 OR BARTYPE=5 THEN INPUT " enter the horizontal sector of the fixed link: ",A
330 IF BARTYPE<>4 THEN GOTO 380
340 A=(A/180)*PI:DIAG=A-(INT(A/PI)*PI)
350 B=(B/180)*PI:BT=S-(INT(B/PI)*PI)
360 C=(C/180)*PI:CT=C-(INT(C/PI)*PI)
370 D=(D/180)*PI:DT=D-(INT(D/PI)*PI)
380 IF BARTYPE<>3 AND BARTYPE<>5 THEN GOTO 480
390 INPUT " enter the vertical sector of the fixed link: ",F
400 INPUT " enter the skew angle: ",KSI=KSI/(2*PI)
410 IF BARTYPE=5 THEN GOTO 430
420 G=(A+B)*COS(KSI)
430 C=(C^2-F^2-G^2+2*F*G*COS(KSI))^.5
470 IF C=0 THEN CIM=0
480 IF BARTYPE<>4 AND (DIAG>BT+CT+DT OR BT>DIAG+CT+DT OR CT>DIAG+BT+DT OR DT>DIAG+BT+CT) THEN GOTO 500
490 GOTO 520
500 CLS:LOCATE 12,20:PRINT "Illegal linkage (impossible to assemble)":LOCATE 23,25:PRINT "Hit any key to continue"
510 A$=IN$():GOTO 130
520 IF CIM=0 THEN FOR LOP=1 TO 8:PIM(LOP)=0:NEXT:GOTO 540
530 GOTO 590
540 P(1)=(A-B-C-D)/2!:P(2)=(A-B-C-D)/2!
550 P(3)=(A+B-C-D)/2!:P(4)=(A+B-C-D)/2!
560 P(5)=(A-B+C+D)/2!:P(6)=(A-B+C+D)/2!
570 P(7)=(A-B-C-D)/2!:P(8)=(A-B-C-D)/2!
580 GOTO 640
590 FOR LOP=1 TO 8:PIC(LOP)=(-(1)^(LOP+1))*C/2!:NEXT
600 P(1)=(A-B-D)/2!:P(2)=P(1)
610 P(3)=(A-B-D)/2!:P(4)=P(3)
620 P(5)=(A-B-D)/2!:P(6)=P(5)
630 P(7)=(A-B-D)/2!:P(8)=P(7)
640 IF BARTYPE>3 THEN GOTO 730
650 A1=P(7)*P(8)-PIM(7)*PIM(8)
660 B1=P(1)*P(2)-PIM(1)*PIM(2)
670 C1=P(3)*P(4)-PIM(3)*PIM(4)
680 D1=-2*COS(KSI)*B*D
690 E1=(P(5)*P(6)-(PIM(5)*PIM(6))
700 F1=-6*B*SIN(KSI)
710 G1=F*D*SIN(KSI):H1=0:J1=0
720 GOTO 800
730 IF BARTYPE<4 THEN GOTO 800
740 A1=(-(P(7)*P(8))-(PIM(7)*PIM(8))):B1=S1:C1=S
750 D1=4*B*SIN(KSI):IF BARTYPE=4 THEN D1=0
760 E1=(P(5)*P(6))-(PIM(5)*PIM(6))
770 F1=0:G1=0:J1=0:IF BARTYPE=4 THEN J1=4*E+B
780 H1=-2*COS(KSI):IF BARTYPE=4 THEN H1=2*B
790 J1=-2*COS(KSI):IF BARTYPE=4 THEN J1=2*B
800 IF BARTYPE<>2 THEN GOTO 870
810 A1=SIN(P(7))*SIN(P(8))
820 B1=SIN(P(1))*SIN(P(2))
830 C1=SIN(P(3))*SIN(P(4))
840 D1=-2*SIN(B)*SIN(D)
850 E1=SIN(P(5))*SIN(P(6))
860 F1=0:G1=0:H1=0:J1=0
870 IF BARTYPE<>1 THEN GOTO 940
880 A1=P(7)*P(8)
890 B1=P(1)*P(2)
900 C1=P(3)*P(4)
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490 INPUT "Enter the first bound of the input angle: ",PHIB(1)
500 INPUT "Enter the second bound of the input angle: ",PHIB(2)
510 INPUT "Enter increment: ",INC:INC=(INC/180)*PI
520 PRINT "Which solution do you need? 1. (A~B~ 2. (A-B)
530 ANS$=INPUT$(1):SOL=VAL(ANS$)
540 IF SOL<1 OR SOL>2 THEN SOUND 1500,1:GOTO 980
550 PHIB(1)=(PHIB(1)/180)*PI
560 PHIB(2)=(PHIB(2)/180)*PI
570 OPEN "O",#1,"ANG.DAT"
580 OPEN "O",#2,"VEL.DAT"
590 OPEN "O",#3,"ACC.DAT"
600 FOA PHI=PHIB(1) TO PHIB(2) STEP INC
610 T=TAN(.5*PHI)
620 UII=-(DI*T*(GI+JI)*T^2+(GI*HI))/((2*(B1+C1*(T^2)+F1*T))
630 IF ((DI*T*(GI利益*JI)*T^2+(GI+HI))^2)<4*(B1*C1*(T^2)+F1*T)*(A1*(T^2)+E1+F1+I1
640 ) THEN U11=0;U12=0;UIM=1:GOTO 1100
650 U12=(((DI*T*(GI+JI)*T^2+(GI+HI))^2)-4*(B1+C1*(T^2)+F1*T)*(A1*(T^2)+E1+F1+I1
660 +I1*T))/>.5/(2*(B1+C1*(T^2)+F1*T))
670 U2)=U11-U12
680 U(1)=U(1)+U(2)
690 N FOR LOP=1 TO 2
700 MI(LOP)=-(I/U(LOP)^2)*(A1*U(LOP)*.5*DI*U(LOP)^2+.5*FI*(U(LOP)^2+I)+(GI+JI
710 )+U(LOP))/.5/(2*(B1*U(LOP)+C1*U(LOP)+2*T^2)+F1*U(LOP)+T^2+.5*DI*U(LOP)+T^2+.5*G1+H1)
720 MI1(LOP)=MI1(LOP)
730 IF S=1 THEN MI1(LOP)=((U(LOP)^2)*MI1(LOP))/2!
740 IF U1M=1 THEN MI1(LOP)=0:MI1(LOP)=0
750 IF MI1(LOP)<3 THEN MI1(LOP)=3
760 IF MI1(LOP)<-3 THEN MI1(LOP)=-3
770 PI=(3*U(LOP)^2+1)*B1+C1*T^2+F1*T)+U(LOP)*((G1+J1)*T^2+2*D1*T+G1+H1)
780 QI=(2*(T^2)+2*C1+T^2)+U(LOP)+.5*DI*U(LOP)+G1+J1)*T
790 RI=((I/U(LOP)^2)*(3*T^2+1)*(A1*U(LOP)^2+2*(G1+J1))*U(LOP))+(F1*
800 U(LOP)^2+1)-DI*U(LOP)+I1)
810 SI=2*(B1*U(LOP)+C1*U(LOP)+T^2+2+5*DI*U(LOP)+.5*G1+H1)
820 M2(LOP)=-(PI*M1(LOP)^2+Q1*M1(LOP)+R1)/S1
830 IF S=1 THEN M22(LOP)=(M1(LOP)^2)*M1(LOP)+M2(LOP))/(1+U(LOP)^2)/2!
840 IF U1M=1 THEN M22(LOP)=0:M22(LOP)=0
850 IF M22(LOP)<3 THEN M22(LOP)=3
860 IF M22(LOP)<-3 THEN M22(LOP)=-3
870 M22(LOP)=M22(LOP)
880 IF S=1 THEN M22(LOP)=(M1(LOP)^2)*U(LOP)+M2(LOP))/(1+U(LOP)^2)/2!
890 IF U1M=1 THEN M22(LOP)=0:M22(LOP)=0
900 IF M22(LOP)<3 THEN M22(LOP)=3
910 IF M22(LOP)<-3 THEN M22(LOP)=-3
920 NEXT LOP
930 WRITE #1,(PHI/PI)*I80!,MI(SOL)
940 WRITE #2,(PHI/PI)*180!,M1(SOL)
950 CLOSE
960 END

UBER EIN RECHNERISCH VORTEILHAFTES MICRO-COMPUTER
RECHENPROGRAMM FUR DIE KINEMATISCHE ANALYSE DER
FUNDAMENTALEN GELENKMECHANISMEN

Zusammenfassung—Aus grund einer algebraischen Korrespondenz zwischen den Bewegungsgleichungen
der ebenen, sphärischen und räumlichen Gelenkvierecke, sowie der ebenen und räumlichen Schubkurbeln,
wird ein rechnerisch vorteilhaftes Methode entwickelt für eine microcomputer kinematische Analyse dieser
Getriebe.